AD-A119 522 SAN DIEGO STATE UNIV CA DEPT OF MATHEMATICAL SCIENCES F/G 12/1
INFERENCE FOR WEIBULL-TYPE POISSON PROCESSES (ESTIMATION AND SI--ETC(U)
AUG 82 S M LEE, C B BELL N00014-80-C-0208
UNCLASSIFIED 5-82 NL

END
DATE
FILMED
11 82
DTIC

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 5-82	2. GOVT ACCESSION NO.: AD-A119522
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): Inference for Weibull-type Poisson Processes (Estimation and Signal Detection)	5. TYPE OF REPORT & PERIOD COVERED: Technical
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): S.M. Lee and C.B. Bell	8. CONTRACT OR GRANT NUMBER(s): N00014-80-C-0208
9. PERFORMING ORGANIZATION NAME AND ADDRESS: San Diego State University, San Diego, CA 92182	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: Office of Naval Research, Arlington, VA 22217	12. REPORT DATE: Aug. 82
	13. NUMBER OF PAGES: 38
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

Approved for public release; distribution unlimited.

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Poisson process, failure rate, mean function, Weibull-distribution, censoring, same-shape and same-distance sampling, waiting time, interarrival, shape parameter, nuisance parameter, likelihood, goodness-of-fit, Kolmogorov-, Lilliefors-, and Srinivasan-type statistics.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Signal detection is studied in the context of Weibull-type non-homogeneous Poisson processes. Inference for the one- and two-parameter cases is treated for four different sampling plans- Type I- and Type II - censoring; same-shape and same-distance sampling. Extensions of the Kolmogorov-Simirnov statistic are employed as well as techniques based on the liklehood function.

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

INFERENCE FOR WEIBULL-TYPE POISSON PROCESSES

(Estimation and Signal Detection)

S. M. Lee and C. B. Bell

San Diego State University

1. Introduction and Summary.

NHPP's (i.e., non-homogeneous Poisson processes) describe a wide variety of physical phenomena- background noise, epileptic seizure patterns, failure patterns etc. A particularly useful NHPP is the Weibull-type NHPP - one for which the mean function is of the form $\mu(t) = \alpha t^{\beta}$. It is immediate that by appropriate choice of β, the shape parameter, one can achieve a constant failure rate, and an increasing failure rate, as well as a decreasing failure rate. Consequently, this family, Ω(WPP), of Weibull-type NHPP's incorporates a wide variety of useful models.

The developments in the sequel will be concerned with interval and point estimation, as well as signal detection, which is, of course, closely related to hypothesis testing.

In deriving the inference procedures, one will consider four different sampling plans, which will be described in detail in the

This research was principally supported by the Office of Naval Research through Grant No. N00014-80-C-0208.

sequel. They are (A) Type I Censoring; (B) Type II Censoring, (C) Same-shape Sampling; and (D) Same-distance Sampling.

It will be found that the requirements, the data, and the feasible types of inference are different for these different plans.

The paper is divided into twelve sections. Some basic definitions and background material is given in Section 2. Section 3 contains a description of the sampling plans. Sections 4 and 5 present the necessary distribution theory.

Detection problems are treated in Sections 6 through 9. Estimation is treated in Sections 10 and 11. Open problems and conclusions are presented in the final section. Finally, numerical illustrations of the methodology developed are presented in the appendix.

2. Some Basic Definitions and Distributions.

The stochastic law, L, of a NHPP is completely determined by the mean function of the process. Mean functions here will be of slightly restricted type.

Definition 2.1. A mean function $\mu(\cdot)$ is a real valued function (defined for non-negative time and) satisfying (i) $\mu(0) = 0$; (ii) $\mu(\infty) = \infty$; (iii) $\mu(\cdot)$ is continuous; and (iv) $\mu(\cdot)$ is strictly increasing.

Some common mean functions are given below.

Example 2.1. (i) For HPP's, i.e., homogeneous Poisson processes, one has $\mu(t) = \lambda t$, for some $\lambda > 0$. (ii) For each law L in Ω(WPP), $\mu(t) = \alpha t^{\beta}$, for some positive α and β. One notes that when $\beta = 1$, one has a HPP. (iii) $\mu(t) = a \ln(1 + bt)$. (iv) $\mu(t) = -\ln[\Phi(-p\ln \lambda t)]$, where $\Phi(\cdot)$ is the standard Gaussian cpf. (v) $\mu(t) = a(e^{\beta t} - 1)$, related to the Gumbel-Extreme-Value density (vi) $\mu(t) = \lambda t \exp\{\underset{\sim}{Z}^t \underset{\sim}{B}\}$, where $\underset{\sim}{Z}$ and $\underset{\sim}{B}$ are vectors. This corresponds to the Cox proportional hazard model

A NHPP can now be formally defined.

Definition 2.2. (a) Consider a counting process $\{N(t): t \geq 0\}$ satisfying (i) $N(0) = 0$, (ii) increments are independent, and (iii) $P\{N(t) - N(s) = k\} = [\mu(t) - \mu(s)]^k \exp\{-[\mu(t) - \mu(s)]\}/k!$ $0 \leq s \leq t$, and $k = 0, 1, 2, \ldots$ where $\mu(\cdot)$ is a mean function. Such a point process is the counting process of a NHPP with mean function $\mu(\cdot)$. (b) $\{W_r: r \geq 1\}$ is the (dual) waiting-time process of the NHPP with mean function $\mu(\cdot)$, if the W's satisfy $\{N(t) \leq n\} = \{W_{n+1} > t\}$ for $n = 0, 1, 2, \ldots$. (c) $\{X_r: r \geq 1\}$ is the (dual) interarrival-time process if $X_r = W_r - W_{r-1}$, where $W_0 = 0$.

There are some special relations between W_1 and $\mu(\cdot)$ which characterize the stochastic law, L, of a NHPP.

Theorem 2.1. Let $F(t) = P\{W_1 \leq t\}$ and $f(t) = \frac{dF}{dt}$. Then (1) $F(0) = 0$ and $F(t) = 1 - \exp\{-\mu(t)\}$, for $t \geq 0$.

(2) $f(t) = \mu'(t) \exp\{-\mu(t)\}$, for $t \geq 0$.

(3) $\mu'(t) = \frac{f(t)}{1-F(t)}$ and

(4) $\mu(t) = -\ln[1 - F(t)]$.

In the sequel, one will be interested solely in $\Omega(WPP)$, the family of Weibull-type NHPP laws L.

Definition 2.3. A NHPP has a law L in $\Omega(WPP)$ if its mean function is of the form $\mu(t) = \alpha t^{\beta}$ $(t \geq 0)$, for positive α and β.

The other relevant functions for $\Omega(WPP)$ are as follows.

Theorem 2.2. For L in $\Omega(WPP)$ with $\mu(t) = \alpha t^{\beta}$, one has

(i) $\mu'(t) = \alpha\beta t^{\beta-1}$, $t \geq 0$,

(ii) $F(t) = 1 - \exp\{-\alpha t\}$, $t \geq 0$; and $f(t) = \alpha\beta t^{\beta-1} \exp\{-\alpha t^{\beta}\}$, $t \geq 0$.

The first waiting time W_1, then, has a (two-parameter) Weibull distribution. This accounts for the name of this particular family of NHPP's.

By an appropriate change of scale and/or an appropriate choice of parameters a WPP can be transformed to a HPP, i.e., homogeneous Poisson process.

Theorem 2.3. Let $\{N(t): t \geq 0\}$ have a law L in $\Omega(WPP)$, i.e., $\mu(t) = \alpha t^{\beta}$ for some positive α and β; let $M(s) = N(s^{1/\beta})$, for $s \geq 0$; let $V_r = W_r^{\beta}$; and $Z_r = V_r - V_{r-1}$, where $V_0 = W_0 = 0$. Then

(1) $\{N(t): t \geq 0\}$ is a HPP iff $\beta = 1$;

(2) $\{M(t): t \geq 0\}$ is a HPP with $\mu(t) = \alpha t$;

(3) $\{V_r\}$ and $\{Z_r\}$ are, respectively, the waiting-times and interarrival times of $\{M(t)\}$; and

(4) $\{Z_r\}$ are i.i.d. $\text{Exp}(\alpha)$.

These insights will be used in constructing the detection procedures in the sequel.

Also, one will employ the Kolmogorov-Smirnov statistic (Kolmogorov, 1933) and its modifications based on the works of Lilliefors (1967, 1969); Srinivasan (1970) and Choi (1980).

Let $\underset{\sim}{X} = (X_1, \ldots, X_n)$ be a random sample with common continuous cpf $F(\cdot;\theta)$; and let $\Omega'' = \{F(\cdot;\theta): \theta \in \text{Ⓗ}\}$. Let Ω'' admit a M-S-S (minimal sufficient statistic), $S(\underset{\sim}{X})$. Further, let $\underset{\sim}{X}^* = [X(1), \ldots, \underset{\sim}{X}(n)]$ be the order statistics of $\underset{\sim}{X}$ and $F_n(\cdot)$ be the empirical cdf of the X's, i.e., $F_n(z) = \frac{1}{n} \sum_{j=1}^{n} \varepsilon\,(z - X_j) = \frac{1}{n} \sum_{1}^{n} \varepsilon\,(z - X(j))$.

The following statistics will be employed in the sequel.

<u>Definition 2.4</u>. Let θ be an arbitrary fixed element of Ⓗ; and $\hat{\theta}$ be the MLE (maximum likelihood estimate) based on $\underset{\sim}{X}$,

(i) $\hat{F}_n(z) = F(z;\hat{\theta})$, for all z.

(ii) $\tilde{F}_n(z) = E\{F_n(z) | S(\underset{\sim}{X})\}$;

(iii) $D_n(\theta) = \sup_z |F_n(z) - F(z;\theta)|$;

(iv) $\hat{D}_n = \sup_z |F_n(z) - \hat{F}_n(z)|$; and

(v) $\tilde{D}_n = \sup_z |F_n(z) - \tilde{F}_n(z)|$.

The cpfs of these last three statistics are utilized in treating signal detection. The notation to be used for these and other relevant distributions is given in Definition 2.5. below.

Definition 2.5. The cpfs K-S(n); Ω'' - LI(n); Ω'' - SR(n); G-O-S(n); U-O-S(n) are defined by the following distributions of statistics.

(1) $D_n(\theta) \sim$ K-S(n);

(2) $\hat{D}_n \sim \Omega''$ - LI(n);

(3) $\tilde{D}_n \sim \Omega''$ - SR(n);

(4) $\underset{\sim}{X}^* \sim$ G-O-S(n), when $G(\cdot) = F(\cdot,\theta_0)$; and

(5) $\underset{\sim}{X}^* \sim$ U-O-S(n), when $G(\cdot) = U(0,1)$.

Some standard distributions are denoted using the notation below.

NOTATION: (1) χ^2_m refers to the chi-square distribution with m degrees of freedom.

(2) $F(m,r)$ refers to the classical F-distribution with m and r degrees of freedom.

(3) $\Gamma(m,\lambda)$ denotes the gamma distribution with parameters m and λ.

(4) $H(\cdot;\gamma,A)$ is the cpf satisfying $H(z;\gamma,A) = (\frac{z}{A})^\gamma$ for $0 < z < A$, with γ and $A > 0$.

(5) $NB(m,p)$ denotes the negative binomial cpf such that $P\{X = r\} = \binom{m+r-1}{r}p^r q^m$ for $r = 0, 1, 2, \ldots$.

One next presents the four sampling plans.

3. The Four Sampling Plans.

For life-testing and survival analysis both in engineering fields and in biomedical testing, censoring arises naturally.

Plan I. Type I Censoring.

One observes the WPP on the interval $[0,T^*]$, and records $[N(T^*);\ W_1, \ldots, W_{N(T^*)}] = [k;\ W_1, \ldots, W_k]$, where $N(T^*) = k \geq 1$.

This sampling plan is used when the time interval available for study is limited. It, of course, could happen that very few events occur in $[0, T^*]$, i.e., k is small. If such an occurrence has a reasonably probability, one might wish to consider an alternate sampling plan.

Plan II. Type II Censoring.

Here one observes the WPP until the kth waiting time, W_k; and records $\underset{\sim}{W} = [W_1, \ldots, W_k]$.

This sampling plan is used when a certain minimum amount of data is needed. It could happen that W_k is much larger than anticipated, and one cannot afford the time. Consequently one might wish to consider some combination of Plans I and II. (Such a combination will not be considered in the sequel).

Plan III. Same-Shape Sampling.

This is a direct adaptation of a sampling plan used by Basawa and Rao (1980) for HPP's.

Let $\{N(t): t \geq 0\}$ be the WPP (of interest) with mean function $\mu(t) = \alpha t^{\beta}$ and waiting times $\{W_n\}$; and let $\{N^*(t): t \geq 0\}$ be an independent WPP with mean function $\mu^*(t) = t^{\beta}$; and waiting times $\{W_n^*\}$. These two mean functions have the same shape, i.e., $[\mu(t)][\mu^*(t)]^{-1}$ is constant. This plan can be implemented only when β is known.

One observes the process of interest only at the waiting times $\{W_n^*\}$. The data is then

$$\underset{\sim}{N} = [N(W_1^*), \ldots, N(W_k^*)] \quad \text{or} \quad \underset{\sim}{Y}^* = [Y_1^*, \ldots, Y_k^*], \quad \text{where}$$

$$Y_r^* = N(W_r^*) - N(W_{r-1}^*) \quad \text{with} \quad W_0^* = 0.$$

Sampling in this fashion yields tractable distributions for some statistics of interest. In fact, one can prove, analagous to Basawa and Rao (1980)

<u>Theorem 3.1</u>. (1) $Y_1^*, \ldots, Y_k^*$ are i.i.d. $NB(1, \frac{1}{1+\alpha})$ and (2) $N(W_k^*) = \sum_1^k Y_j \sim N - B(k, \frac{1}{1+\alpha})$.

<u>Plan IV</u>. <u>Same-Distance Sampling</u>.

One observes the process $\{N(t)\}$ at times $0 < t_1 < t_2 < \ldots < t_k$, where $\mu(t_r) = r\mu(t_1)$. This means that $t_r = t_1(r^{1/\beta})$, for $r = 1, 2, \ldots, k$; and that one must know β, to implement the plan.

The data here is $\underset{\sim}{N} = [N(t_1), \ldots, N(t_k)]$ or $\underset{\sim}{Y} = [Y_1, \ldots, Y_k]$ where $Y_r = N(t_r) - N(t_{r-1})$ with $t_0 = 0$. The Y's, in this case,

constitute a random sample, and, hence, some of the statistics to be used will have more tractable distributions.

The distribution theory of interest is given below.

4. Distribution Theory when the Shape Parameter, β, is known

When β is known, the same-shape and same-distance samplin .s, which will be described in the sequel, are best handled by transforming the WPP to a HPP (as in Theorem 2.3), with mean function $\mu(t) = \alpha t$. Inference, then concerns only one parameter (and only HPP's).

For Type I and Type II censoring, one might wish to bring in some different types of results.

(A) Type I Censoring.

Here one observes the process on the time interval $[0,T^*]$, and receives data

$$[N(T^*);\ W_1, W_2, \ldots, W_{N(T^*)}] = [k;\ W_1, \ldots, W_k] \quad \text{when}$$

$$N(T^*) = k \geq 1.$$

For such data one has the following useful distribution theorem for $\Omega(\text{WPP})$.

Theorem 4.1. Conditionally, given $N(T^*) = k(\geq 1)$,

(i) $[W_1, \ldots, W_k] \sim$ G-O-S(k), where $G(\cdot) = H(\cdot,\beta,T^*)$ (See Section 3).

(ii) $[(\frac{W_1}{T^*})^\beta, \ldots, (\frac{W_k}{T^*})^\beta] \sim$ U-O-S(k); and

(iii) $[Y_1, Y_2, \ldots, Y_k] \sim$ Exp(β)-O-S(k), where

$$Y_j = \ln(\frac{W_{k+1-j}}{T^*}) \quad \text{for} \quad 1 \leq j \leq k.$$

These order-statistic distributions lead naturally to the K-S (and other goodness-of-fit) statistics.

Theorem 4.2. Conditionally, given $N(T^*) = k(\geq 1)$, each of the following statistics is distributed K-S(k).

(i) $\sup_z |\frac{1}{k} \sum_1^k \varepsilon(z - W_j) - (\frac{z}{T^*})^\beta|$,

(ii) $\sup_{0<u<1} |\frac{1}{k} \sum_1^k \varepsilon(u - [\frac{W_j}{T^*}]^\beta) - u|$

(iii) $\sup_z |\frac{1}{k} \sum_1^k \varepsilon(z - Y_j) - [1 - e^{-z\beta}]|$

For Type II censoring, the two theorems above change only slightly.

(B) Type II Censoring

Here one observes the WPP until the kth waiting time W_k; and the data is $\underset{\sim}{W} = [W_1, \ldots, W_k]$.

Theorem 4.3. Conditionally, given $W_k = w^*$,

(i) $[W_1, \ldots, W_{k-1}] \sim$ G-O-S(k - 1), where $G(\cdot) = H(\cdot, \beta, w^*)$;

(ii) $[(\frac{W_1}{w^*})^\beta, \ldots, (\frac{W_{k-1}}{w^*})^\beta] \sim$ U-O-S(k - 1); and

(iii) $[Y_1, Y_2, \ldots, Y_k] \sim \text{Exp}(\beta)\text{-O-S}(k - 1)$, where

$$Y_j = -\ln\left(\frac{W_{k-j}}{w^*}\right), \quad \text{for} \quad 1 \leq j \leq k - 1.$$

The relevant K-S distributions are, then, as given below.

Theorem 4.4. Conditionally, given $W_k = w^*$, each of the following statistics is distributed K-S $(k - 1)$.

(i) $\sup\limits_{z} \left| \frac{1}{k-1} \sum\limits_{1}^{k-1} \varepsilon\,(z - W_j) - \left(\frac{z}{w^*}\right)^{\beta}\right|$;

(ii) $\sup\limits_{0<u<1} \left| \frac{1}{k-1} \sum\limits_{1}^{k-1} \varepsilon\,\left(u - \left[\frac{W_j}{w^*}\right]^{\beta}\right) - u\right|$; and

(iii) $\sup\limits_{z} \left| \frac{1}{k-1} \sum\limits_{1}^{k-1} \varepsilon\,(z - Y_j) - [1 - e^{-z\beta}]\right|$.

If both α and β are known, one can utilize

Theorem 4.5. $2\mu(W_k) = 2\alpha W_k^{\beta} \sim \chi^2_{2k}$.

When β is unknown, the problems are significantly different. As a nuisance parameter, the shape parameter β cannot be eliminated in the same way one would eliminate the scale parameter α.

5. Distribution Theory when β is unknown

When the shape parameter is unknown, one can still construct a HPP from the data, but the development is more involved.

One starts with Type I Censoring and Theorem 4.1, in which $Y = [Y_1, \ldots, Y_k] \sim \text{Exp}(\beta)\text{-O-S}(k)$, where

$$Y_j = -\ln\left[\frac{W_{k+1-j}}{T^*}\right].$$ Now let

$$Z_r^* = \sum_{j=1}^{r} (k+1-j)Y_j = -\sum_{m=k+1-r}^{k} m \ln\left(\frac{W_m}{T^*}\right)$$ and

$$\underset{\sim}{E} = \{E_1, E_2, \ldots, E_{k-1}\} = \left\{\frac{Z_r^*}{Z_k^*};\ 1 \le r \le k-1\right\}.$$

One can then prove

<u>Theorem 5.1</u>. *Conditionally, given,* $N(T^*) = k$,

(1) $Z_r^* \sim \Gamma(r,\beta)$ and $2\beta Z_r^* \sim \chi^2_{2r}$ for $r = 1, 2, \ldots, k$;

(2) $\underset{\sim}{E} \sim \text{U-O-S}(k-1)$; and

(3) Z_k^* and $\underset{\sim}{E}$ are independent.

Since neither $\underset{\sim}{E}$ nor its distribution involves the (nuisance) parameters α and β, it can be used to investigate questions of the structure of the process.

For the two-sample situation, one "eliminates" β in an analagous fashion.

Consider two processes. The first $\{N(t)\}$ has law L_1 and waiting times $\underset{\sim}{W} = (W_1, \ldots, W_m)$ in the interval $[0, T^*]$; and the second process $\{N^*(t)\}$ with law L_2, has in the same time interval waiting time $\underset{\sim}{V^*} = (V_1^*, \ldots, V_n^*)$. Now let

$$Y_j = -\ln\left(\frac{W_{m+1-j}}{T^*}\right) \quad \text{for} \quad 1 \le j \le m, \quad \text{and}$$

$$U_j = -\ln\left(\frac{V^*_{n+1-j}}{T^*}\right) \quad \text{for} \quad 1 \le j \le n.$$

Then, one readily proves

Theorem 5.2. Conditionally, given $N_1(T^*) = m$ and $N_2(T^*) = n$, $\frac{\overline{Y}}{\overline{U}} \sim F(2m,\ 2n)$, if L_1 and L_2 are in $\Omega(\text{WPP})$ and $\beta_1 = \beta_2$.

One notes here that α does not explicitly appear in the results above. However, the value of α does influence the distributions of $\{N(t)\}$ and $\{N^*(t)\}$

For Type II Censoring, the results are quite similar.

Theorem 5.3. (i) Conditionally, given $W_{k+1} = w^*$, $\underset{\sim}{E}$ and $\{Z_r\}$ have the distributions in Theorem 5.1 if w^* replaces T^* in the definition of Y_j.

(ii) Conditionally, given $W_{m+1} = w_1^*$ and $V_{n+1} = v^*$, the conclusion of Theorem 5.2 is valid, when

$$Y_j = -\ln\left(\frac{W_{m+1-j}}{w^*}\right) \quad \text{and} \quad U_j = -\ln\left(\frac{V_{n+1-j}}{v^*}\right).$$

One can now treat the signal detection problems.

6. Signal Detection: Goodness-of-fit with Two Nuisance Parameters.

The signal detection problem here is as given below.

$\underline{PN_1}$: $L \in \Omega(\text{WPP})$ vs. $\underline{N + S_1}$ $L \notin \Omega(\text{WPP})$

Based on Theorems 4.1 and 5.1 and Definitions 2.4 and 2.5, one can construct the following decision rules.

Decision Rule 6.1. For Sampling Plan I, i.e., Type I Censoring decide $\underline{N + S_1}$ iff

$$D_{k-1} = \sup_{\mathcal{U}} \left|\frac{1}{k-1} \sum_{1}^{k-1} \varepsilon\,(\mathcal{U} - E_j) - \mathcal{U}\right| > d(\alpha,k-1), \quad \text{where}$$

$d(\alpha, k-1)$ is the appropriate percentile of the K-S(k - 1), distribution, and the E's are as in Section 5.

Decision Rule 6.2. For Type I Censoring, decide $\underline{N + S_1}$ iff

$$\hat{D}_k = \sup_{z} \left|\frac{1}{k} \sum_{1}^{k} \varepsilon\,(z - Y_j) - [1 - \exp\{-\frac{z}{\bar{Y}}\}]\right| > \hat{d}(\alpha,k)$$

where $\hat{d}(\alpha,k)$ is the appropriate percentile of the Ω''-LI distribution of Definition 2.5 and Lilliefors (1969), with $\Omega'' = \{\mathrm{Exp}(\beta):\ \beta > 0\}$.

Decision Rule 6.3. For Type I Censoring, decide $\underline{N + S_1}$ iff

$$\tilde{D}_k = \sup \left|\frac{1}{k} \sum_{1}^{k} \varepsilon\,(z - Y_j) - [1 - (1 - \frac{z}{k\bar{Y}})^{k-1}]\right| > \tilde{d}(\alpha,k)$$

where $\tilde{d}(\alpha,k)$ is the appropriate percentile of the Ω''-SR(k) distribution of Definition 2.5 and Srinivasan (1970), with $\Omega'' = \{\mathrm{Exp}(\beta):\ \beta > 0\}$.

[Note: In Section 11, it will be seen that the $\bar{Y}$, used in the two decision rules above, is the MLE of β.]

Similar decision rules exist for Type II Censoring and for other goodness-of-fit statistics. These rules will not be given here. The given rules are illustrated by numerical examples in the appendix.

The other detection problem with two nuisance parameters to be considered here is the two-sample problem.

7. Signal Detection: A Two-Sample Problem.

Here, one is comparing two sets of data, and the PN-situation is the equality of the two Weibull-type laws.

$$\underline{PN_2}: \quad L_1 = L_2 \qquad \text{vs.} \qquad \underline{N + S_2}: \quad L_1 \neq L_2$$

The decision rule here will be given for the case of Type II Censoring only. However, the corresponding rule for Type I Censoring is very similar and can be easily derived.

Consider data $\underset{\sim}{Z} = [W_1, \ldots, W_m, V_1, \ldots, V_n]$ and Y's and V's as in Theorem 5.3.

Decision Rule 7.1. Decide $\underline{N + S_2}$ iff $\frac{\bar{Y}}{\bar{U}} < f'$ or $> f''$, where f' and f'' are the appropriate percentiles of an $F(2m, 2n)$ distribution.

The latter decision rule is based on the minimal sufficient statistics of the Y-sample and U-sample and is presumed to be optimal. Therefore no other decision rule will be considered for this case.

The next detection problem involves no nuisance parameters.

8. Signal Detection: Goodness-of-fit.

One has

$\underline{PN_3}$: $L \in \Omega(WPP)$ and $(\alpha,\beta) = (\alpha_0, \beta_0)$ vs

$\underline{N + S_3}$: Not $\underline{PN_3}$

Decision Rule 8.1. Decide $\underline{N + S_3}$ iff $N(T^*) \leq C_1$ or $\geq C_2$ or if $C_1 < k = N(T^*) < C_2$ and

$$D(k) = \sup_z \left|\frac{1}{k} \sum_1^k \varepsilon \;(z - Y_j) - [1 - e^{-\beta_0 z}]\right| > d(\alpha,k_2)$$

where $Y_j = -\ln \frac{W_{k+1-j}}{T^*}$ and C_1 and C_2 are appropriate percentiles of a Poisson distribution with mean $\nu = \alpha_0(T^*)^{\beta_0}$.

The decision rule above involves both a "size" statistics $N(T^*)$ and a shape statistic $D(k)$. The PN-distribution of $N(T^*)$, the total number of waiting times, is Poisson with parameter $\nu = \alpha_0[T^*]^{\beta_0}$ and does not involve the distribution of waiting times within the interval $[0, T^*]$. On the other hand, the statistic $D(k)$ depends directly on the distribution of the k waiting times, and is hence a "shape" statistic.

These statistics are used separately in the one-parameter problems discussed below.

9. Signal Detection: One-parameter Problems.

There are two basic one-parameter detection problems for Ω(WPP).

$\underline{PN_4}$: $\beta = \beta_0$ vs. $\underline{N + S_4}$: $\beta \neq \beta_0$; and

$\underline{PN_5}$: $\alpha = \alpha_0$ (assuming β, known) vs. $\underline{N + S_5}$: $\alpha \neq \alpha_0$.

For the first problem above, one utilizes the statistic $D(k)$ of Decision Rule 8.1. For the second problem above, one employs the statistic $N(T^*)$ of the same decision rule.

It is interesting to note that one need not know α, in order to examine questions about β; but the reverse is not true.

One now turns to estimation problems, where the emphasis will be on MLE's, unbiased point estimates and confidence intervals.

10. Estimation when the Shape Parameter, β, is known.

Theorem 10.1. The likelihood functions for the four sampling plans are as follows. (See Section 3 for a description of the plans and associated data.)

(1) Plan I: $L(k; w_1, \ldots, w_k) = (\alpha\beta)^k (\prod_1^k w_j)^{\beta-1} \exp\{-\alpha(T^*)^\beta\}$.

(2) Plan II: $L(w_1, \ldots, w_k) = (\alpha\beta)^k (w_k)^{\beta-1} (\prod_1^{k-1} w_j)^{\beta-1} \exp\{-\alpha(w_k)^\beta\}$.

(3) Plan III: $L(y_1, \ldots, y_k) = (\prod_1^k y_j!)^{-1} \exp\{-k\alpha t_1^\beta\} \cdot \{\alpha t_1^\beta\}^{\sum_1^k y_j}$,

where $Y_j = N(t_j) - N(t_{j-1})$; $\sum_1^k Y_j = N(t_k)$ and $t_k = t_1(k)^{1/\beta}$.

(4) Plan IV: $L(y_1, \ldots, y_k) = (\frac{\alpha}{\alpha+1})^{\sum_1^k y_j} (\frac{1}{\alpha+1})^k$, where

$Y_j = N(w_j^*) - N(w_{j-1}^*)$.

Now let $\hat{\alpha}$ be the MLE for the plan under consideration; and $\tilde{\alpha}$ be the associated unbiased estimate of α.

These statistics and their distributions are given below.

Theorem 10.2. The MLE's, unbiased estimates and sampling distributions for the four plans are as given in Table 10.1 below.

TABLE 10.1. Estimating α when β is known.

(MLE's; unbiased Estimates and Distributions)

Sampling Plan	$\hat{\alpha}$, the MLE	$\tilde{\alpha}$, the associated unbiased estimate	Relevant Distribution
I (Type I Censoring)	$\frac{N(T^*)}{(T^*)^\beta}$	$\hat{\alpha}$	$(T^*)^\beta\hat{\alpha} \sim P_0(\nu)$, with $\nu = \alpha(T^*)^\beta$
II (Type II Censoring)	$\frac{k}{(W_k)^\beta}$	$(\frac{k-1}{k})\hat{\alpha}$	$\frac{2\alpha k}{\hat{\alpha}} \sim \chi^2_{2k}$
III (Same-Shape Sampling)	$\frac{N(W_k^*)}{k}$	$\hat{\alpha}$	$k\hat{\alpha} \sim NB(k, p)$ where $p = \frac{1}{\alpha+1}$
IV (Same-Distance Sampling)	$\frac{N(t_k)}{(t_k)^\beta} = \frac{N(t_1 k^{1/\beta})}{t_1^\beta k}$	$\hat{\alpha}$	$kt_1^\beta\hat{\alpha} = t_k^\beta\hat{\alpha} \sim P_0(\nu)$, with $\nu = \alpha t_k^\beta$

Confidence interval for α can now be constructed using the relevant distributions of Table 10.1

Definition 10.1 Let $X \sim \chi^2_m$; $N^* \sim P_0(\nu)$; and $U^* \sim N - B(r,p)$. Let $[b_1(m), b_2(m)]$, $[C_1(N^*), C_2(N^*)]$, and $[d_1(r,p), d_2(r,p)]$ be defined by $1 - \gamma = P\{b_1(m) \leq X \leq b_2(m)\} = P\{C_1(N^*) \leq \nu \leq C_2(N^*)\} = P\{d_1(\underline{U}^*,\underline{r}) \leq p \leq d_2(\underline{U}^*,\underline{r})\}$

[Note: For the c-values, see Rao et.al.(1966); and for the d-values, see Clemans (1959)].

The confidence intervals of level $(1 - \gamma)$ for α can be constructed as in the theorem below.

Theorem 10.3. The following are respective confidence intervals of levels $(1 - \gamma)$ for α with the indicated sampling plans.

(1) PLAN I: (Type I Censoring)

$$\frac{C_1(N(T^*))}{(T^*)^\beta} < \alpha < \frac{C_2(N(T^*))}{(T^*)^\beta}$$

(2) PLAN II: (Type II Censoring)

$$\frac{\hat{\alpha} b_1(2k)}{2k} < \alpha < \frac{\hat{\alpha} b_2(2k)}{2k}$$

(3) PLAN III: (Same-Shape)

$$\frac{d_1(N(w_k^*),k)}{1-d_1(N(w_k^*),k)} < \alpha < \frac{d_2(N(w_k^*),k)}{1-d_2(N(w_k^*),k)}$$

(4) PLAN IV: (Same-distance)

$$\frac{C_1(N(t_k))}{(t_k)^\beta} < \alpha < \frac{C_2(N(t_k))}{(t_k)^\beta}$$

One notes that $N(T^*)$ and $N(t_k)$ are discrete random variables, and, hence, only a discrete set of confidence levels is attainable for Plans I and IV.

Numerical examples illustrating these techniques are presented in the appendix.

When β is not known, the estimation problems are quite different. In fact, Plans III and IV cannot be implemented, and, hence, one considers only Type I and Type II censoring.

11. Estimation when β is unknown

For Type I censoring, one can prove

Theorem 11.1. (1) The MLE's of α and β satisfy the equations

$$\hat{\alpha} = \frac{N(T^*)}{(T^*)^{\hat{\beta}}} \quad \text{and} \quad \hat{\beta} = \frac{N(T^*)}{N(T^*)\ln T^* - \sum_1^k \ln W_j}$$

(2) $$\frac{2\beta N(T^*)}{\hat{\beta}} \sim \chi^2_{2k} .$$

Recalling the definitions of the previous section, one has

Theorem 11.2.

$$P\left\{\frac{b_1(2k)\hat{\beta}}{2N(T^*)} < \beta < \frac{b_2(2k)\hat{\beta}}{2N(T^*)}\right\} = 1 - \gamma$$

This confidence interval does not depend on α. Further, there is no simple tractable confidence interval available for α.

For Type II censoring, it can be proved that

Theorem 11.3. (1) The MLE's of α and β satisfy the equations

$$\hat{\alpha} = \frac{k}{(W_k)^{\hat{\beta}}} \quad \text{and} \quad \hat{\beta} = \frac{k}{(k-1)\ln W_k - \sum_{1}^{k-1} W_j} \text{ ; and}$$

(2) $\frac{2k\beta}{\hat{\beta}} \sim \chi^2_{2(k-1)}$

The resulting confidence interval for β is given by

Theorem 11.4. $P\left\{ \frac{b_1(2(k-1))\hat{\beta}}{2k} < \beta < \frac{b_2(2(k-1))\hat{\beta}}{2k} \right\} = 1 - \gamma.$

Again, no tractable confidence interval for α is available in this case.

The final section discusses conclusions and open problems.

12. Concluding Remarks; Open Problems

(A) Applications. It is clear that Ω(WPP) contains a wide variety of NHPP's, and, therefore can be used to model a variety of types of data. From the problems posed and solved, it is also clear that Ω(WPP) is a rich source of problems - both theoretical and applied.

(B) Other Sampling Plans. Only two sampling plans were considered when β is unknown. It is desirable to develop other reasonable plans, since in applications β will rarely be known. Regularly spaced sampling yields data of the form $\underset{\sim}{N} = [N(\Delta), N(2\Delta), \ldots, N(k\Delta)]$. But the MLE's, etc., are totally intractable.

(C) Optimal Procedures. For each of the inference problems considered, one or more reasonable procedures is given. Are these procedures optimal? Should goodness-of-fit statistics other than the K-S statistics be used. For example, is the Cramer-von-Mises statistic better here?

(D) Confidence Intervals for α. When β is unknown, the authors were unable to construct reasonable confidence intervals for α. Would another type of sampling plan yield better results in this direction?

(E) Joint Confidence Regions. Does the distribution theory presented allow one to construct reasonable confidence regions for the pair (α,β)? This, for example, is readily done for certain Gaussian processes.

(F) Weibull Fits. In the body of the text a goodness-of-fit test is given for a WPP. However, in seeking to model engineering or biomedical data, etc., with a WPP, it would be interesting to develop a set of axioms which the underlying physical forces should satisfy in order to generate a WPP. Such axiom systems are common for HPP's, for example. Again, one asks about the feasibility of using goodness-of-fit tests other than the K-S.

REFERENCES

1. Basawa, I. V., and Rao, B. L. S. P. (1980). Statistical Inference for Stochastic Processes. Academic Press, London.

2. Bell, C. B. (1964). Automatic Distribution-Free Statistical Signal Detection, Research Report #1245 (October). U.S. Navy, Electronics Laboratory.

3. Bell, C. B., Woodroofe, M. F. and Avadhani, T. V. (1970). "Nonparametric Tests for Stochastic Processes," in Nonparametric Techniques in Statistical Inference, ed. Madan Lal Puri. Cambridge University Press.

4. Bickel, P. J., and Doksum, K. A. (1977). Mathematical Statistics, San Francisco, Holden-Day, Inc.

5. Choi, Y. J. (1980). The Kolmogorov-Smirnov Test with Nuisance Parameters in the Uniform Case. M. S. Thesis, University of Washington.

6. Çinlar, E. (1975). Introduction to Stochastic Processes, Prentice Hall, Inc.

7. Clemans, Kermit G. (1959). "Confidence limits in the case of the geometric distribution", Biometrika, 46, 260-264.

8. Cox, D. R. and P. A. Lewis (1966). The Statistical Analysis of a Series of Events, London: Methuen.

9. Durbin, J. (1973). Distribution Theory for Tests Based on the Sample Distribution Function, SIAM, CBMS-NSF Regional Conference.

10. Kalbfleisch, J. D. and Prentice R. L. (1980). The Statistical Analysis of Failure Time Data, John Wiley and Sons, Inc.

11. Kendall, M. G., and Stuart, A. (1973). The Advanced Theory of Statistics, Vol. 2, 3rd ed., Hafner Publishing Co., New York.

12. Lilliefors, H. W. (1967). "On the Kolmogorov-Smirnov test for normality with mean and variance unknown", J. Amer. Stat. Assn., 62, 399-402.

13. Lilliefors, H. W. (1969). "On the Kolmogorov-Smirnov test for the exponential distribution with mean unknown", J. Amer. Stat. Assn., 64, 387-389.

14. Rao, C. R; Mitra, S. K. and Matthai, A. (1966). Formulae and Tables for Statistical Work, Statistical Publishing Society, Calcutta.

15. Parzen, Emanuel (1962). Stochastic Processes, Holden-Day, San Fransisco.

16. Rao, C. R.; Mitra, S. K.; and Matthai, A. (1966), Formulae and Tables for Statistical Work., Statistical Publishing House, Calcutta.

17. Srinivasan, R. (1970). "An approach to testing goodness-of-fit of incompletely specified distributions", Biometrika 57(3), 605-611.

APPENDIX: NUMERICAL EXAMPLES

In the appendix one is concerned with three sets of data in illustrating the methodology developed. Table A.1. contains EKG-data for 25 cycles of a "normal" heart. Table A.2. contains simulated WPP data with several sets of parameter values. In the numerical examples, Data Sets 2 and 3 are used (as well as the EKG-data).

(I) Numerical Examples for Section 6. There are three decision rules in this section, and they are illustrated using the data of Table A.1. Modifications of the data to be used with the various decision rules of Section 6 are given in Tables A.2, A.3, and A.5.

TABLE A.1. EKG-Data

Cycle No	MAGNITUDE P	Q	R	T	TIMES P	Q	R	T	Cycle Length
1	.0927	.0708	1.190	.4785	148	428	467	743	1014
2	.1005	.0917	1.203	.4344	1,183	1,440	1,484	1,758	1014
3	.1121	.0982	1.212	.4316	1,897	2,378	2,498	2,884	1040
4	.1238	.0608	1.297	.5652	3,452	3,521	3,558	3,835	1041
5	.0819	.0206	1.261	.5060	4,108	4,559	4,599	4,812	1056
6	.0769	.0764	1.284	.4797	5,346	5,618	5,655	5,923	1078
7	.0815	.0507	1.228	.4700	6,346	6,689	6,733	7,005	1073
8	.1263	.1113	1.166	.4240	7,596	7,756	7,806	8,085	1076
9	.1009	.0817	1.274	.4542	8,645	8,840	8,882	9,154	1089
10	.0839	.0601	1.239	.4663	9,694	9,935	9,971	10,250	1044
11	.1126	.0907	1.164	.4462	10,727	10,985	11,015	11,301	1012
12	.0939	.0894	1.189	.4554	11,833	11,985	12,027	12,300	1028
	.0929	.0648	1.278	.4778	12,876	13,017	13,055	13,331	1038
14	.1010	.0555	1.213	.4742	13,758	14,050	14,093	14,367	1062
15	** .0560	.0374	1.235	.4874	14,307	15,112	15,153	15,429	1063
16	.0832	.0899	1.294	.4268	16,007	16,179	16,216	16,495	1083
17	.1278	.0939	1.294	.4465	17,053	17,265	17,299	17,587	1063
18	.1044	.0585	1.255	.5001	18,255	18,325	18,362	18,360	1018
19	.1184	.0702	1.250	.4491	19,286	19,345	19,380	19,655	1013
20	.1142	.1141	1.266	.4105	20,203	20,345	20,393	20,665	947
21	.1338	.1140	1.228	.4052	21,103	21,297	21,340	21,612	891
22	.1331	.1050	1.230	.4405	22,035	22,195	22,231	22,495	911
23	** .0849	.0584	1.127	.4265	23,068	23,097	23,140	23,415	901
24	.0916	.0807	1.156	.4038	23,825	24,002	24,041	24,308	947
25	.1442	.1323	1.260	--	24,766	24,937	24,988	--	

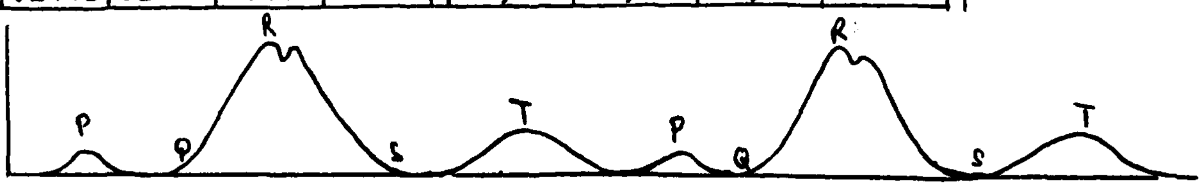

TABLE A.2. Illustration of Decision Rule 6.1.

	X_j	W_j	Y_j	Z_r	E_r	$E(j)-\frac{j-1}{k-1}$	$\frac{j}{k-1}-E(j)$
1	1014	1014	0.0200	0.46	0.0035	0.0035	0.0441
2	1014	2028	0.0590	1.758	0.0133	-	0.0776
3	1040	3068	0.1000	3.858	0.0291	-	0.1073
4	1041	4109	0.1419	6.696	0.0506	-	0.1312
5	1056	5165	0.1883	10.2737	0.0776	-	0.1497
6	1078	6243	0.2406	14.6045	0.1103	-	0.1624
7	1073	7316	0.2960	19.6365	0.1484	-	0.1698
8	1076	8392	0.3573	25.3533	0.1915	-	0.1721
9	1089	9481	0.4239	31.7118	0.2396	-	0.1695
10	1044	10525	0.4938	38.625	0.2918	-	0.1627
11	1012	11537	0.5690	46.022	0.3477	-	0.1523
12	1028	12565	0.6484	53.8028	0.4065	-	0.1390
13	1038	13603	0.7337	61.8735	0.4675	-	0.1234
14	1062	14665	0.8255	70.1285	0.5298	-	0.1066
15	1063	15728	0.9300	78.4985	0.5931	-	0.0887
16	1083	16811	1.0520	86.9145	0.6566	-	0.0707
17	1063	17874	1.1892	95.2389	0.7195	-	0.0532
18	1018	18892	1.3478	103.3257	0.7806	0.0079	0.0376
19	1013	19905	1.5374	111.0127	0.8387	0.0205	0.0249
20	947	20852	1.7661	118.0771	0.8921	0.0285	0.0170
21	891	21743	2.0583	124.252	0.9387	0.0296	0.0158
22	911	22654	2.4723	129.1966	0.9761	0.0216	0.0239
23	901	23555	3.1654	132.362			
24	947	24502					

24 Data points $T^* = 24030$, $k = 23$

Let $\{X_j\}$ be the interarrivals and $\{W_j\}$ be the waiting times.

Also let $Y_j = -\ln\left(\frac{W_{k+1-j}}{T^*}\right)$, $Z_r = \sum_{j=1}^{r} (k+1-j)Y_j = -\sum_{m=k+1-r}^{k} m \ln\left(\frac{W_m}{T^*}\right)$

and $E_r = \frac{Z_r}{Z_k}$.

The critical value (on interpolating) is 0.254. D_{k-1} is computed to be 0.1721. Therefore, one decides PN.

TABLE A.3. Illustration of Decision Rule 6.2.

	Y_j	$* = 1-\exp\{-\frac{Y(j)}{\bar{Y}}\}$	$* - \frac{j-1}{23}$	$\frac{j}{23} - *$
1	0.0200	0.0221	0.0221	0.0214
2	0.0590	0.0637	-	0.0280
3	0.1000	0.1056	0.0186	0.0248
4	0.1419	0.1464	0.0160	0.0275
5	0.1883	0.1895	0.0156	0.0279
6	0.2406	0.2354	0.0180	0.0255
7	0.2960	0.2812	0.0203	0.0231
8	0.3573	0.3288	0.0245	0.0190
9	0.4239	0.3768	0.0290	0.0145
10	0.4938	0.4236	0.0323	0.0112
11	0.5690	0.4700	0.0155	0.0083
12	0.6484	0.5149	0.0366	0.0068
13	0.7337	0.5589	0.0372	0.0063
14	0.8255	0.6019	0.0367	0.0068
15	0.9300	0.6457	0.0370	0.0065
16	1.0520	0.6908	0.0386	0.0049
17	1.1892	0.7347	0.0390	0.0044
18	1.3478	0.7777	0.0386	0.0049
19	1.5374	0.8201	0.0375	0.0060
20	1.7661	0.8606	0.0345	0.0090
21	2.0583	0.8994	0.0298	0.0136
22	2.4723	0.9366	0.0236	0.0229
23	3.1654	0.9707	0.0142	0.0293

For this data

$\bar{Y} = 0.89634$ and $\hat{D}_k = 0.0390$.

Interpolation in the Lilliefors (1969) table yields a critical value of 0.22. The decision is therefore: PN.

TABLE A.4. Illustration of Decision Rule 6.3.

	Y_j	$* = 1-(1-\frac{Y(j)}{k\bar{Y}})^{k-1}$	$* - \frac{j-1}{23}$	$\frac{j}{23} - *$
1	0.0200	0.0211	0.0211	0.0224
2	0.0590	0.0611	0.0176	0.0259
3	0.1000	0.1015	0.0145	0.0279
4	0.1419	0.1410	0.0106	0.0329
5	0.1883	0.1828	0.0089	0.0346
6	0.2406	0.2276	0.0102	0.0333
7	0.2960	0.2725	0.0116	0.0318
8	0.3573	0.3193	0.0150	0.0285
9	0.4239	0.3669	0.0191	0.0244
10	0.4938	0.4134	0.0221	0.0214
11	0.5690	0.4598	0.0250	0.0185
12	0.6484	0.5049	0.0266	0.0168
13	0.7337	0.5494	0.0277	0.0158
14	0.8255	0.5930	0.0278	0.0157
15	0.9300	0.6378	0.0291	0.0144
16	1.0520	0.6841	0.0319	0.0116
17	1.1892	0.7294	0.0337	0.0097
18	1.3478	0.7741	0.0350	0.0085
19	1.5374	0.8182	0.0356	0.0079
20	1.7661	0.8606	0.0345	0.0090
21	2.0583	0.9011	0.0315	0.0119
22	2.4723	0.9398	0.0268	0.0167
23	3.1654	0.9744	0.0179	0.0256

One computes $\bar{Y} = 0.89634$

$\tilde{D}_k = 0.0356$ and the interpolated Srinivasan value is 0.15. Therefore, one decides: PN.

(II) Numerical Examples for Section 7.

The decision rules of Section 7 are illustrated with data from Table A.5.

TABLE A.5. Simulated Waiting Times with $\mu(t) = \alpha t^{\beta}$

	Data Set 1 $\alpha = 4.0$ $\beta = 0.5$	Date Set 2 $\alpha = 4.0$ $\beta = 1.0$	Date Set 3 $\alpha = 2.5$ $\beta = 2.0$	Data Set 4 $\alpha = 1.0$ $\beta - 2.0$
1	0.1794	0.2640	0.6621	1.5370
2	0.2118	0.3535	0.8156	1.8415
3	3.5143	0.8270	1.1033	2.2670
4	3.5301	1.4313	1.1425	2.4329
5	4.8938	1.7190	1.3875	2.6718
6	5.1777	2.3334	1.4569	2.9192
7	5.2139	2.7005	1.6535	2.9655
8	5.4513	2.9981	1.6554	3.3051
9	5.5830	3.0349	1.7187	3.3359
10	5.8101	3.4578	1.8768	3.6561
11	6.6792	3.8193	2.1242	3.6623
12	7.1624	4.1066	2.1813	3.8465
13	7.1868	4.2841	2.2706	3.8716
14	8.3186	4.2980	2.3851	3.9012
15	10.1834	4.3277	2.4161	4.4260
16	17.7497	4.4119	2.5016	4.4830
17	20.9384	4.9939	2.5161	4.6552
18	37.8407	5.7613	2.5183	4.7408
19	42.9353	6.1637	2.5231	4.8441
20	47.5891	6.4948	2.5745	4.8968
21	47.6528	6.6561	2.7927	4.9904
22	48.2519	7.1748	2.9971	5.0636
23	48.8106	7.2400	3.0902	5.0779
24	49.7105	7.3192	3.1346	5.0933
25	53.5510	7.4199	3.1652	5.2479
26	54.3202	7.7230	3.1985	5.4329
27	55.3261	7.7686	3.2867	5.4493
28	56.2949	7.9864	3.3588	5.4559
29	67.6298	8.3356	3.4849	5.5021
30	72.0625	9.4819	3.5187	5.6157
31	72.1961	9.7305	3.5476	5.6721
32	75.2956	10.1198	3.5925	5.7419
33	78.5233	10.3810	3.8160	5.9278
34	79.6223	10.5817	3.8490	5.9652
35	81.8531	10.8142	3.9115	6.0453
36	89.8449	10.9833	3.9896	6.1611
37	95.4882	11.1056	4.0754	6.2578
38	100.9558	11.3623	4.1004	6.3962
39	108.3140	11.6609	4.1185	6.3982
40	115.3873	11.7117	4.1229	6.4602
41	122.6267	12.1708	4.2126	6.5531
42	139.9208	12.9102	4.2949	6.5856
43	142.7151	12.9625	4.2966	6.5952
44	146.4252	13.6038	4.3525	6.6012
45	146.8417	14.5538	4.4296	6.6321
46	148.9848	15.2888	4.4329	6.7129
47	205.9331	15.3994	4.4472	6.7171
48	211.0332	15.4766	4.5433	6.9320
49	217.9563	15.5336	4.5572	7.0581
50	239.4034	15.5897	4.5681	7.1849
51	250.9719	16.3864	4.5883	7.2028

TABLE A.6. Illustration of Decision Rule 7.1.

[Here one uses simulated Data Set 2 with α = 4.0, β = 1.0]

The first 24 points constitute the first sample; and the next 20 points constitute the second sample.

1st Sample			2nd Sample		
	W_j	Y_j		V_j	U_j
1	0.2640	0.0137	1	0.3031	0.1429
2	0.3535	0.0245	2	0.3487	0.2524
3	0.8270	0.0336	3	0.5665	0.2619
4	0.4313	0.1086	4	0.9157	0.4065
5	0.1790	0.1332	5	2.062	0.5082
6	2.3334	0.1855	6	2.3106	0.5201
7	2.7005	0.2530	7	2.6999	0.5931
8	2.9981	0.3959	8	2.9611	0.6604
9	3.0349	0.5199	9	3.1618	0.6941
10	3.4578	0.5391	10	3.3943	0.7428
11	3.8193	0.5460	11	3.5634	0.8137
12	4.1066	0.5493	12	3.6857	0.8793
13	4.2841	0.5916	13	3.9424	0.9716
14	4.2980	0.6641	14	4.241	1.1204
15	4.3277	0.7635	15	4.2918	1.2412
16	4.4119	0.8940	16	4.7509	2.0529
17	4.9939	0.9062	17	5.4903	2.5331
18	5.7613	1.0107	18	5.5426	3.0184
19	6.1637	1.1568	19	6.1839	3.1586
20	6.4948	1.4624	20	7.1339	
21	6.6561	1.6456			
22	7.1748	2.1941			
23	7.2400	3.0440			
24	7.3192	3.3360			
25	7.4199				

One computes $V^* = 7.1339$, $W^* = 7.4199$, $\overline{Y} = 0.8738$, $\overline{U} = 1.0831$ and $\frac{\overline{Y}}{\overline{U}} = 0.8068$. The interpolated F-values for 48 and 38 degrees of freedom are 0.5473 and 1.8676. Therefore, one decides: PN.

(III) Numerical Examples for Sections 8 and 9.

The Decision Rule 8.1 is a combination of two decision rules. The application of these two rules is given below in the context of Section 9 and the EKG-Data of Table A.1.

One first considers

$$\underline{PN_4}: \quad \underset{(\beta_0=2)}{\beta = \beta_0} \quad \text{vs.} \quad \underline{N + S_4}: \quad \beta \neq \beta_0$$

The statistics: $D(k) = \sup_z \left| \frac{1}{k} \sum_1^k \varepsilon(z - Y_j) - [1 - e^{-\beta_0 z}] \right|$

where $Y_j = -\ln \left[\frac{W_{k+1-j}}{T^*}\right]$, $(T^* = 11031)$.

TABLE A.7. Illustration relative to PN_4 of Section 9.

	X_j	W_j	Y_j	$* = 1-e^{-\beta_0 z}$	$* - \frac{j-1}{10}$	$\frac{j}{10} - *$
1	1014	1014	0.0470	0.0897	0.0897	0.0103
2	1014	2028	0.1514	0.2613	0.1613	-
3	1040	3068	0.2734	0.4212	0.2212	-
4	1041	4109	0.4106	0.5601	0.2601	-
5	1056	5165	0.5692	0.6797	0.2797	-
6	1078	6243	0.7588	0.7808	0.2808	-
7	1073	7316	0.9875	0.8612	0.2612	-
8	1076	8392	1.2797	0.9226	0.2226	-
9	1089	9481	1.6937	0.9662	0.1662	-
10	1044	10525	2.3868	0.9916	0.0916	0.0084

One computes $D(k) = 0.2808$, and finds the critical value to be $= 0.369$. Therefore, one decides PN.

For the next illustration, consider

$$\underline{PN_5}: \quad \alpha = .072 \quad \text{vs.} \quad \underline{N + S_5}: \quad \alpha \neq .072$$

(Assuming β known to be 0.4)

Decision Rule. Decide N + S iff

$$N(\ *) \geq 8 \quad \text{or} \quad \leq 0$$

Here the FAR $\approx$ .05

We know that $N(T^*) \overset{PN_5}{\sim} P_0(\nu)$ where $\nu = \alpha(T^*)^\beta = (.072)(11031)^{.4} =$ 2.98.

From the data, $N(T^*) = 10$. Therefore, one decides $\underline{N + S_5}$.

(IV) <u>Numerical Examples for Section 10</u>.

The data is that of Data Set 3 of Table A.6.

(A) The point estimates from Theorem 10.2 are as follows:

<u>PLAN I</u>: $\hat{\alpha} = \tilde{\alpha} = \dfrac{N(T^*)}{(T^*)^\beta} = \dfrac{50}{(4.58)^2} = 2.384$

(Here, take $T^* = 4.58$ and $\beta = 2$. From the data one sees $N(T^*) = 50$).

<u>PLAN II</u>: (a) $\hat{\alpha} = \dfrac{k}{(W_k)^\beta} = \dfrac{51}{(4.5833)^2} = 2.4278$, when one chooses $k = 51$.

(b) $\tilde{\alpha} = \dfrac{k-1}{k}\hat{\alpha} = (\frac{50}{51})(2.4278) = 2.3802$

<u>PLAN III</u>: $\hat{\alpha} = \tilde{\alpha} = \dfrac{N(W_k^*)}{k} = \dfrac{47}{16} = 2.9375$.

Here, one chooses $k = 16$; and generated the first 16 waiting times $\{W_n^*\}$ of a WPP with $\mu(t) = t^2$. It turns out that for this second process, which is independent of the process of interest, $W_{16}^* = 4.4830$ and $N(W_{16}^*) = 47$.

PLAN IV: $\hat{\alpha} = \tilde{\alpha} = \dfrac{N(t_k)}{(t_k)^\beta} = \dfrac{50}{(4.58)^2} = 2.384$ when one chooses

$t_k = 4.58$ and $k = 16$. This means that $t_1 = (t_k)(k)^{-1/\beta} = 1.145$ and $t_j = t_1(j)^{1/\beta} = (1.145)\sqrt{j}$ for $j = 1, 2, \ldots, 16$.

(B) The Confidence Intervals based on Theorem 10.3.

PLAN I: (Type I Censoring). Here $\beta = 2.0$ is known; $T^* = 4.58$, $k = 50$ from Data Set 3 and one seeks a 95% confidence interval for α. One has $N(T^*) \sim P_0(\nu)$, where $\nu = \alpha(T^*)^\beta = \alpha(4.58)^2$. From the table on p. 46 of Rao et.al.(1966), one finds

$$\left[\frac{C_1(N(T^*))}{(T^*)^\beta}, \quad \frac{C_2(N(T^*))}{(T^*)^\beta}\right] \text{ leads to}$$

$$\frac{37.67}{(4.58)^2} = 1.7958 < \alpha < 3.0963 = \frac{64.95}{(4.58)^2}$$

PLAN II: (Type II Censoring). Here $k = 51$, $W_k = 4.5833$ and $\beta = 2.0$ is known; and one needs the X^2-percentiles for 102 degrees of freedom

$$\left[\frac{\hat{\alpha}b_1(2k)}{2k}, \quad \frac{\hat{\alpha}b_2(2k)}{2k}\right] \text{ leads to}$$

$$\frac{(2.4278)(76.25)}{(2)(51)} = 1.8149 < \alpha < 3.1547 = \frac{(2.4278)(132.54)}{2(51)}$$

PLAN III: (Same-Shape) Here one uses Data Set 3, and an independent WPP with mean function $\mu(t) = t^{\beta} = t^2$. For $k = 16$, $W_k^* = 4.4830$ and $N(W_k^*) = 47$. Since $N(W_k^*)$ has a negative binomial distribution, one employs the methodology and graphs of Clemans (1959), to construct a 90% confidence interval for the mean, m^*, of this negative binomial distribution, based on $\hat{m}^* = \frac{N(W_k^*)}{k} = \frac{47}{16} = 2.9375$. The 90% band on m^* from Clemans graphs is $2.1 < m^* < 5.2$. But in terms

of α, $m^* = \frac{\frac{\alpha}{\alpha+1}}{\frac{1}{\alpha+1}} = \alpha$, and hence $2.1 < \alpha < 5.2$.

PLAN IV (Same-Distance) Here one again uses Data Set 3; assumes $\beta = 2$ is known; and chooses $t_1 = 1.07$. Then $t_j = (t_1)(j)^{1/\beta} = 1.07 \sqrt{j}$, for $j = 1, 2, \ldots, 16$

$$\frac{C_1(N(t_k))}{(t_k)^{\beta}} = 1.7958 < \alpha < 3.0963 = \frac{C_2(N(t_k))}{(t_k)^{\beta}}$$

One now considers the cases when β is unknown.

(V) Illustrations of the Methodology of Section 11.

PLAN I. (Type I Censoring) Here one uses Data Set I of Table A.6, and $T^* = 245$. (The formulae are from Theorems 11.1 and 11.2).

(a) Point Estimates: The MLE's are $\hat{\beta} = \frac{N(T^*)}{N(T^*)\ln T^* - \sum_{1}^{k} \ln W_j} = 0.4894$.

and $\hat{\alpha} = \frac{N(T^*)}{(T^*)^{\beta}} = \frac{50}{(245)^{0.4894}} = 3.386$.

(b) 95% Confidence Interval for β.

$$\frac{b_1(2k)\hat{\beta}}{2N(T^*)} = 0.3648 < \beta < 0.6325 = \frac{b_2(2k)\hat{\beta}}{2N(T^*)}$$

where the X^2-percentiles (b's) are approximately 74.55 and 129.24 for 100 degrees of freedom.

PLAN II (Type II Censoring) (The formulae are from Theorems 11.3 and 11.4). One uses Data Set I and k = 5, from which $W_{51} = 250.9719$.

(a) Point Estimates. One easily calculates

$$\hat{\beta} = \frac{k}{(k-1)\ln W_k - \sum_1^{k-1} W_j} = 0.4934 \quad \text{and}$$

$$\hat{\alpha} = \frac{k}{(W_k)^{\hat{\beta}}} = \frac{51}{(250.9719)^{0.4934}} = 3.273.$$

(b) 95% Confidence Interval for β.

The b's are as in the illustration of PLAN I, i.e., $b_1(100) = 74.55$ and $b_2(100) = 129.24$, interpolating from a X^2-table. Hence

$$\frac{b_1(100)\hat{\beta}}{2k} = 0.3606 < \beta < 0.6252 = \frac{b_2(100)\hat{\beta}}{2k}$$

This completes the numerical examples.

DATE

ILME